Name of College - S.S. College, J. Bad
Deptt - Mathematics
Topic - Integration of irrational Algebraic Functions
Class - B.Sc I (Hons)
Time — 10:15 A.M To 11:00 A.M Date - 27-08-2020
11:00 A.M To 11:45 A.M
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Form $\int \frac{1}{L\sqrt{M}}\,dx$ where L and M are both linear

Method
Put $\sqrt{M} = z$ i.e. $M = z^2$

Problem $\int \frac{dx}{(2+x)\sqrt{1+x}}$

Put $\sqrt{1+x} = z$
$\Rightarrow 1+x = z^2$
$x+2 = x+1+1$
$= z^2+1$
$dx = 2z\,dz$

$$= \int \frac{2\cancel{z}\,dz}{(z^2+1)\cancel{z}}$$
$$= 2\int \frac{dz}{1+z^2} = 2\tan^{-1} z + C \qquad \left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

Putting $z = \sqrt{x+1}$
$$= 2\tan^{-1}\sqrt{x+1} + C$$

2. $\int \frac{dx}{(2x+1)\sqrt{4x+3}}$

Put $z = \sqrt{4x+3} \Rightarrow z^2 = 4x+3$
$z^2-1 = 4x+2$
$= 2(2x+1)$
$\Rightarrow \frac{z^2-1}{2} = 2x+1$
Also $2z\,dz = 4\,dx$
$\Rightarrow \frac{z\,dz}{2} = dx$

$$= \frac{1}{\cancel{2}}\int \frac{\cancel{z}\,dz}{\frac{(z^2-1)}{\cancel{2}}\cancel{z}}$$
$$= \int \frac{dz}{z^2-1} \qquad \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$$
$$= \frac{1}{2}\log\frac{z-1}{z+1} + C = \frac{1}{2}\log\frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1} + C$$

2. Form : $\int \frac{1}{L\sqrt{M}}\,dx$

Where L is Linear
and M is quadratic

Step :- put $L = 1/z$

Ex :-> $\int \frac{dx}{(x-1)\sqrt{x^2+1}}$

put $x-1 = \frac{1}{z} \Rightarrow x = \frac{1}{z}+1$

$dx = -\frac{1}{z^2}\,dz$

$x^2+1 = \left(\frac{1}{z}+1\right)^2+1 = \frac{2z^2+2z+1}{z^2}$

$\therefore \int \frac{dx}{(x-1)\sqrt{x^2+1}} = \int \frac{-\frac{1}{\cancel{z^2}}\,dz}{\frac{1}{\cancel{z}}\frac{\sqrt{2z^2+2z+1}}{\cancel{z}}}$

$= -\int \frac{dz}{\sqrt{2z^2+2z+1}}$

$= -\int \frac{dz}{\sqrt{2\left(z^2+z+\frac{1}{2}\right)}}$

$= -\frac{1}{\sqrt{2}}\int \frac{dz}{(z+1/2)^2+\frac{1}{2}-\frac{1}{4}}$

$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\left(z+\frac{1}{2}\right)^2+\frac{1}{4}}$

$= -\frac{1}{\sqrt{2}} \sinh^{-1}\frac{z+\frac{1}{2}}{1/2} + C$

$= -\frac{1}{\sqrt{2}} \sinh^{-1}(2z+1) + C$

$= -\frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{x+1}{x-1}\right) + C$

$\because z = \frac{1}{x-1}$

$2z = \frac{2}{x-1}$

$2z+1 = \frac{2}{x-1}+1$

$= \frac{2+x-1}{x-1}$

$= \frac{x+1}{x-1}$

Problem 2. $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

~~Solu~~

Put $1+x = \frac{1}{z}$ So that

$dx = -\frac{dz}{z^2}$ $\quad z = \frac{1}{1+x}$

$x = \frac{1}{z} - 1$

$x^2 = \left(\frac{1}{z} - 1\right)^2 = \frac{1}{z^2} - 2\cdot\frac{1}{z} + 1$

$1 - x^2 = \cancel{1} - \frac{1}{z^2} + \frac{2}{z} - \cancel{1}$

$= \frac{2z-1}{z^2}$

$\therefore I = \int \frac{dx}{\frac{1}{z}\frac{\sqrt{2z-1}}{z}} = -\int \frac{1}{\frac{\sqrt{2z-1}}{\cancel{z^2}}}\frac{dz}{\cancel{z^2}}$

$= -\int (2z-1)^{-1/2} dz = -\frac{(2z-1)^{-1/2+1}}{(-1/2+1)\times 2}$

$\left[\text{Using } \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}\right]$

$= -(2z-1)^{1/2} + C$

$= -\left[\frac{2}{1+x} - 1\right]^{1/2} + C$

$= -\left[\frac{2-1-x}{1+x}\right]^{1/2} + C$

$= -\sqrt{\frac{1-x}{1+x}} + C$

Problem 3. $\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

Put $1+x = \frac{1}{z}$

$$\Rightarrow dx = -\frac{1}{z^2}\,dz$$

Also, $1+2x-x^2 = 1+2\left(\frac{1}{z}-1\right) - \left(\frac{1}{z}-1\right)^2$

$$= 1 + \frac{2}{z} - 2 - \left[\frac{1}{z^2} - \frac{2}{z} + 1\right]$$

$$\cancel{= 1 + \frac{2}{z} - 2 - \frac{1}{z^2} + \frac{2}{z} - 1}$$

$$= \cancel{1} + \frac{2}{z} - 2 - \frac{1}{z^2} + \frac{2}{z} - \cancel{1}$$

$$= \frac{2z - 2z^2 - 1 + 2z}{z^2}$$

$$= \frac{4z - 2z^2 - 1}{z^2}$$

$$\therefore \int \frac{dx}{(1+x)\sqrt{1+2x-x^2}} = -\int \frac{\frac{dz}{z^2}}{\frac{1}{z}\frac{\sqrt{4z-2z^2-1}}{z}}$$

$$= -\int \frac{dz}{\sqrt{4z-2z^2-1}}$$

$$= -\int \frac{dz}{\sqrt{-2\left[z^2-2z+\frac{1}{2}\right]}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{-z^2+2z-\frac{1}{2}}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{\frac{1}{2}-(z-1)^2}}$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1} \frac{z-1}{\frac{1}{\sqrt{2}}}$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1}\left\{\sqrt{2}(z-1)\right\} + C$$

Putting $z = \frac{1}{1+x}$

$$z - 1 = \frac{1}{1+x} - 1$$

$$= \frac{1-1-x}{1+x}$$

$$= -\frac{x}{1+x}$$

Since $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} x/a$.

$$= -\frac{1}{\sqrt{2}} \sin^{-1} \left\{ -\sqrt{2} \frac{x}{1+x} \right] + C$$

$\sin^{-1}(-x) = -\sin^{-1} x$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \frac{\sqrt{2}x}{1+x} + C$$

Problem $\int \frac{dx}{(1+x)\sqrt{1+3x+3x^2}}$

Put $1+x = \frac{1}{z}$ so that $dx = -\frac{dz}{z^2}$

$$\therefore x = \frac{1}{z} - 1$$

$$I = -\int \frac{\frac{dz}{z^2}}{\frac{1}{z}\sqrt{1+3\left(\frac{1}{z}-1\right)+3\left(\frac{1}{z}-1\right)^2}}$$

$$= -\int \frac{dz}{\frac{1}{z}\sqrt{1+\frac{3}{z}-\cancel{3}+\frac{3}{z^2}-\frac{6}{z}+\cancel{3}}}$$

$$= -\int \frac{dz / \cancel{z^2}}{\frac{1}{\cancel{z}}\sqrt{z^2+3z+3-6z}}$$

$$= -\int \frac{dz}{\sqrt{z^2-3z+3}}$$

$$= \int \frac{dz}{\sqrt{(z-3/2)^2 + (\sqrt{3}/2)^2}}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left[\frac{x+\sqrt{x^2+a^2}}{a}\right]$$

$$= -\log \frac{z-3/2+\sqrt{(z-3/2)^2+(\sqrt{3}/2)^2}}{\sqrt{3}/2} + C$$

$$= -\log \frac{(2z-3)+2\sqrt{z^2-3z+3}}{\sqrt{3}} + C$$

$$= -\log \left[\frac{2}{1+x} - 3 + 2\sqrt{\left(\frac{1}{1+x}\right)^2 - 3\cdot\frac{1}{1+x} + 3}\right] + C$$

$$= -\log \frac{2\sqrt{3x^2+3x+1} - (3x+1)}{\sqrt{3}(3x+1)} + C$$

Problem

$$\int \frac{dx}{x\sqrt{3x^2+2x+1}}$$

put $x = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2}dz$

$$\therefore \int \frac{dx}{x\sqrt{3x^2+2x+1}} = \int \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{3\cdot\frac{1}{z^2}+2\cdot\frac{1}{z}+1}}$$

$$= -\int \frac{dz}{\sqrt{3+2z+z^2}}$$

$$= -\int \frac{dz}{\sqrt{(z+1)^2-(\sqrt{2})^2}} = -\sin^{-1}\frac{z+1}{\sqrt{2}}$$

$$= -\sin^{-1}\frac{\frac{1}{x}+1}{} = -\sin^{-1}\frac{x+1}{\sqrt{2}x}$$

Form :→ $\frac{1}{L\sqrt{M}}$, where DATE: 27-08-2020

L, M both are quadratic

Put $\sqrt{\frac{M}{L}} = z$

or $\frac{M}{L} = z^2$

$$= \int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$$

put $\frac{1-x^2}{1+x^2} = z^2$ So that $1-x^2 = z^2(1+x^2)$

$$\Rightarrow \frac{1}{z^2} = \frac{1+x^2}{1-x^2}$$

$$\frac{1+z^2}{1-z^2} = \frac{1+\cancel{x^2}+1-\cancel{x^2}}{\cancel{1}+x^2-\cancel{1}+x^2}$$

$$= \frac{2}{2x^2} = \frac{1}{x^2}$$

$$\Rightarrow x^2 = \frac{1-z^2}{1+z^2}$$

$$2x\,dx = \frac{-2z(1+z^2)-(1-z^2)2z}{(1+z^2)^2}dz$$

$$= \frac{-4z}{(1+z^2)^2}dz$$

$$\therefore dx = -\frac{2z}{(1+z^2)^2}dz$$

Also $1+x^2 = 1+\frac{1-z^2}{1+z^2} = \frac{2}{1+z^2}$

and $1-x^2 = 1-\frac{1-z^2}{1+z^2} = \frac{2z^2}{1+z^2}$

$$\therefore \int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = -2\int \frac{z\,dz}{(1+z^2)^2}\cdot\frac{\sqrt{1+z^2}}{\sqrt{1-z^2}}\cdot\frac{1+z^2}{2}\times\frac{\sqrt{1+z^2}}{\sqrt{2}z}$$

$$= -\cancel{2}\int \frac{dz}{\cancel{2}\sqrt{2}\sqrt{1-z^2}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dz}{\sqrt{1-z^2}} = -\frac{1}{\sqrt{2}}\sin^{-1} z = -\frac{1}{\sqrt{2}}\sin^{-1}\sqrt{\frac{1-x^2}{1+x^2}}$$